Banach spaces: fundamental theorems problem set

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1. Let $(X, \|\cdot\|)$ be a normed space.

   a) *(Riesz lemma)* Assume M is a proper closed subspace of X. Let $0 < \theta < 1$ be given. Show that there is an $x \in X$ such that $\|x\| = 1$ and $\|x - z\| \geq \theta$ $(z \in M)$.

   b) Let B be the closed unit ball of X, $B = \{x \in X : \|x\| \leq 1\}$. Prove that B is compact if, and only if, X is finite-dimensional.

2. Let $(X, \|\cdot\|)$ be a normed space. Show that:

   a) There exist a Banach space $\widetilde{X}$ and an isometry $T : X \to T(X) \subset \widetilde{X}$ such that $T(X)$ is a dense subspace of $\widetilde{X}$.

   b) The space $\widetilde{X}$ is unique up to isometries.

   [*Remark:* $\widetilde{X}$ is called the *completion* of X.]

3. Let X_i be normed vector spaces, with norms $\|\cdot\|_i$ $(1 \leq i \leq n)$. The cartesian product space $X_1 \times X_2 \times \cdots \times X_n = \prod_{i=1}^n X_i$ is defined by

$$\prod_{i=1}^{n} X_i = \{(x_1, x_2, \ldots, x_n) : x_i \in X_i \ (1 \leq i \leq n)\}.$$

   In $\prod_{i=1}^n X_i$ we consider coordinatewise addition:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \quad (x_i, y_i \in X_i,\ 1 \leq i \leq n)$$

   and coordinatewise scalar multiplication:

$$\lambda\,(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n) \quad (\lambda \in \mathbb{K},\ x_i \in X_i,\ 1 \leq i \leq n).$$

   The space $\prod_{i=1}^n X_i$ is endowed with the norm

$$\|(x_1, x_2, \ldots, x_n)\| = \sum_{i=1}^{n} \|x_i\|_i.$$

   Show that:

   a) $(\prod_{i=1}^n X_i, \|\cdot\|)$ is indeed a normed vector space.

   b) If all the spaces $(X_i, \|\cdot\|_i)$ $(1 \leq i \leq n)$ are Banach spaces, then so is $(\prod_{i=1}^n X_i, \|\cdot\|)$.

4. In ℓ^∞ we consider the sets U_1 and U_2, where $U_1 = c_{00}$ consists of all scalar sequences with only finitely many nonzero terms and U_2 is the set of scalar sequences with all but the N first elements equal to zero.

   a) Are U_1 and/or U_2 closed subspaces of ℓ^∞?

   b) Are U_1 and/or U_2 finite dimensional?

5. Consider in ℓ^p $(1 \leq p \leq \infty)$ the subspace c_{00} consisting of all sequences which are eventually zero.

   a) If $1 \leq p < \infty$, is c_{00} dense in ℓ^p?

   b) Is c_{00} dense in ℓ^∞?

6. Prove that

   a) ℓ^p $(1 \leq p < \infty)$ is separable, but

   b) ℓ^∞ is not separable.

7. The space $C^1[a,b] \subset C[a,b]$ can be endowed with the sup-norm

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)| \quad (f \in C[a,b]).$$

   Show that:

   a) Taking $\sup_{t\in(a,b)} |f(t)|$ results in an equivalent norm on $C[a,b]$.

   b) $\left(C^1[a,b], \|\cdot\|_\infty\right)$ is not a Banach space.

   c) The map given by

$$\|f\|_\infty^* = \sup_{t\in[a,b]} |f(t)| + \sup_{t\in[a,b]} \left|f'(t)\right|$$

   is also a norm on $C^1[a,b]$ which endows it with a Banach space structure.

8. *(Subspace topology)* Let X be a topological space, and let $Y \subset X$. Recall that the *subspace topology* is defined on Y by declaring open all the sets of the form $U \cap Y$, where U is open in X. Denote by Cl_X and Int_X the closure and interior with respect to X, and by Cl_Y and Int_Y the closure and interior with respect to the subspace topology on Y. Prove that:

   a) $F \subset Y$ is closed in Y if, and only if, there exists a closed set G in X such that $F = G \cap Y$.

   b) $\mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$ $(A \subset Y)$.

   c) $\mathrm{Int}_Y(A) = \mathrm{Int}_X(A \cup (X \setminus Y)) \cap Y$ $(A \subset Y)$.

9. Let X be a metric space. Show that every subset of a nowhere dense set in X is nowhere dense in X.

10. Let X be a metric space. Prove that the boundary of an open or closed set $A \subset X$ is nowhere dense. Show by counterexample that this conclusion may not hold if A is neither open nor closed.

11. Prove that a subset A of a metric space X is nowhere dense if, and only if, for every nonempty open set $U \subset X$ there exists a nonempty open set $V \subset U$ such that $V \cap A = \emptyset$.

12. Show that the union of a finite number of nowhere dense sets is nowhere dense.

13. Let X be a metric space and let $A \subset Y \subset X$. Prove the following:

a) If A is nowhere dense in the subspace topology of Y, then A is nowhere dense in X.

b) Conversely, if Y is open (or dense) in X and A is nowhere dense in X, then A is nowhere dense in the subspace topology of Y.

Show that the conclusion in *b*) may fail if Y is neither open nor dense in X.

14. Let X be a complete metric space.

a) Prove that if $A \subset X$ is nowhere dense and $G \subset X$ is open, then there exists a closed ball $B \subset G$ such that $B \cap A = \emptyset$. [Cf. Exercise 11.] Moreover, given any $k > 0$, B can be chosen so that $\delta(B) < k$.

b) Deduce from *a*) the following weak form of the Baire category theorem: *Every complete metric space $X \neq \emptyset$ is of the second category*.

15. Show the equivalence of the following statements:

a) Every complete metric space has the Baire property.

b) Every complete metric space is of the second category.

16. Consider the metric space $\mathbb{Q}$ endowed with the subspace topology inherited from that of $\mathbb{R}$.

a) Explain why $\mathbb{Q}$ is not complete with this topology.

b) Prove that every open set in $\mathbb{Q}$ is of the first category in $\mathbb{Q}$. Therefore, noncomplete metric spaces may contain open subsets of the first category.

17. *a*) Let (U_1, d_1), (U_2, d_2) be metric spaces such that $U_1 \cap U_2 = \emptyset$. Define d on $(U_1 \cup U_2) \times (U_1 \cup U_2)$ by:

$$d(x,y) = \begin{cases} 1, & x \in U_1 \text{ and } y \in U_2, \text{ or } x \in U_2 \text{ and } y \in U_1 \\ d_1(x,y), & x, y \in U_1 \\ d_2(x,y), & x, y \in U_2. \end{cases}$$

Prove that $(U_1 \cup U_2, d)$ is a metric space, called the *disjoint union* of (U_1, d_1) and (U_2, d_2), and denoted by $U_1 \sqcup U_2$.

b) Let Y be any complete metric space, and let X be the disjoint union $Y \sqcup \mathbb{Q}$, where $\mathbb{Q}$ is endowed with its usual topology, inherited from that of $\mathbb{R}$. Show that X is a metric space of the second category without the Baire property.

18. Show that a normed space is Baire if, and only if, it is of the second category.

19. Prove the following statements.

a) Let $A \subset B \subset C$ be three sets in a topological space. If A is nowhere dense (resp. of the first category) in B, then A is nowhere dense (resp. of the first category) in C.

b) *(Incomplete normed space of the second category)* Every infinite-dimensional Banach space X contains a dense hyperplane which is of the second category in itself.

20. *(Incomplete normed space of the second category)* Let $\{x_n\}_{n=1}^{\infty}$ be a linearly independent sequence in a Banach space. Show that there exists a non-closed subspace $Y \subset X$ which is of the second category in X (hence in itself) and contains at most finitely many terms of $\{x_n\}_{n=1}^{\infty}$.

21. Prove the following assertions:

    a) Proper subspaces of normed spaces have empty interior.

    b) No infinite-dimensional Banach space can have a countable Hamel basis.

22. Let X be a Baire space. Show that:

    a) The intersection of any countable family of G_δ dense subsets of X is a G_δ dense subset of X.

    b) If G is a G_δ dense subset of X then, with the topology inherited from X, G is a Baire space.

23. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function satisfying the following: for every complex number z, there exists $n = n(z) \in \mathbb{N}$ such that $f^{(n)}(z) = 0$. Does this imply that f must be a polynomial?

24. Suppose X is a Banach space, Y is a normed linear space, and $\{\Lambda_\alpha\}_{\alpha \in A}$ is a collection of bounded linear transformations from X to Y, where A is some index set. Prove that either there exists an $M > 0$ such that

$$\|\Lambda_\alpha\| \le M \quad (\alpha \in A),$$

or

$$\sup_{\alpha \in A} \|\Lambda_\alpha x\| = \infty$$

for all x belonging to some dense G_δ in X. [*Remark:* In geometric terminology, the alternatives are as follows: either there is a ball B in Y (with radius M and center at 0) such that every Λ_α maps the unit ball of X into B, or there exist $x \in X$ (in fact, a whole dense G_δ of them) such that no ball in Y contains $\Lambda_\alpha x$ for all α.]

25. Assume X is a Banach space, Y a normed space, and $\mathscr{A}$ a family of bounded linear operators from X to Y. Set

$$B = \{x \in X : \sup\{\|Tx\| : T \in \mathscr{A}\} < \infty\}.$$

Prove the equivalence of the following statements:

a) B is of the second category in X.

b) $B = X$, that is, $\mathscr{A}$ is pointwise bounded.

c) $\mathscr{A}$ is uniformly bounded: there exists $M > 0$ such that $\|T\| \le M$ for all $T \in \mathscr{A}$.

26. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of bounded linear maps from X to Y, where both X and Y are Banach spaces. Show that a necessary and sufficient condition for $\lim_{n\to\infty} T_n x$ to exist for each $x \in X$ is:

    a) $\lim_{n\to\infty} T_n x$ exists for all x in a dense subset of X, and

    b) $\{\|T_n\|\}_{n=1}^{\infty}$ is bounded.

27. Let $\{\Lambda_n\}_{n=1}^{\infty}$ be a sequence of linear functionals defined on $C[0,1]$, of the form

$$\Lambda_n f = \sum_{k=0}^{N_n} A_{n,k} f(t_{n,k}) \quad (f \in C[0,1]),$$

where, for each $n \in \mathbb{N}$, $\{t_{n,k}\}_{k=0}^{N_n}$ is a finite family of points in $[0,1]$ (called *nodes* of Λ_n). The sequence $\{\Lambda_n\}_{n=1}^{\infty}$ is said to be a *quadrature method* provided that

$$\int_0^1 f(t)\,dt = \lim_{n\to\infty} \Lambda_n f \quad (f \in C[0,1]).$$

Show that:

a) For each $n \in \mathbb{N}$, Λ_n is a continuous linear functional on $C[0,1]$, with norm $\|\Lambda_n\| = \sum_{k=0}^{N_n} |A_{n,k}|$.

b) The sequence $\{\Lambda_n\}_{n=1}^{\infty}$ is a quadrature method if, and only if, the two following conditions hold:

i) $\lim_{n\to\infty} \Lambda_n(x^{k-1}) = 1/k \quad (k \in \mathbb{N})$, and

ii) $\sup_{n\in\mathbb{N}} \sum_{k=0}^{N_n} |A_{n,k}| < \infty$.

c) If $A_{n,k} \geq 0$ for every n and k, then *i*) implies *ii*).

28. Let $\{\alpha(n)\}_{n=1}^{\infty}$ be a sequence of complex numbers, and let $1 \leq p \leq \infty$. Let q be the exponent conjugate to p, that is, q satisfies $1 \leq q \leq \infty$ and

$$\frac{1}{p} + \frac{1}{q} = 1,$$

with the convention that $1/\infty = 0$ (thus, 1 and ∞ are conjugate exponents). Assume that the series $\sum_{n=1}^{\infty} \alpha(n)\xi(n)$ converges for every sequence $\{\xi(n)\}_{n=1}^{\infty} \in \ell^p$. Prove that then $\{\alpha(n)\}_{n=1}^{\infty} \in \ell^q$.

29. As a consequence of the uniform boundedness principle, it can be proved that if E is a Banach space and F a normed space, then every separately continuous bilinear map $B : E \times E \to F$ is jointly continuous. By means of the following counterexample, show that this might not be the case when E is not complete: take for E the linear space of all real polynomials $p = p(t)$, equipped with the norm

$$\|p\|_1 = \int_0^1 |p(t)|\,dt \quad (p \in E),$$

and for B the bilinear functional defined by

$$B(p,q)=\int_0^1 p(t)q(t)\,dt \quad (p,q\in E).$$

30. Let X, Y be pre-Hilbert spaces, and assume that the maps $T:X\to Y$, $S:Y\to X$ satisfy $\langle Tx,y\rangle=\langle x,Sy\rangle$ $(x\in X,\ y\in Y)$. Prove the following statements:

    a) S, T are linear.

    b) If $X=H$ is Hilbert, then S, T are continuous.

    c) If, further, $Y=K$ is Hilbert, then $S=T^*$.

31. Let X be a Banach space. Recall that a sequence $\{x_n\}_{n=1}^{\infty}$ is called a *Schauder basis* for X if to each $x\in X$ there correspond unique scalars $a_n(x)$ $(n\in\mathbb{N})$ such that

$$x=\sum_{n=1}^{\infty} a_n(x)x_n,$$

where the series converges in the norm of X. It can be shown (this may be taken for granted) that $a_n\in X'$ for each $n\in\mathbb{N}$.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis for a Banach space X and $\{y_n\}_{n=1}^{\infty}$ is a Schauder basis for a Banach space Y. Prove that the following two statements are equivalent:

    a) There exists a continuous linear bijection $S:X\to Y$ such that $Sx_n=y_n$ for each $n\in\mathbb{N}$.

    b) Given scalars c_n $(n\in\mathbb{N})$, $\sum_{n=1}^{\infty} c_nx_n$ converges in X if, and only if, $\sum_{n=1}^{\infty} c_ny_n$ converges in Y.

32. Let T be a bounded operator from a Banach space X to a normed space Y such that T is not onto, but $\mathscr{R}(T)$ is dense in Y. Prove that $\mathscr{R}(T)$ is of the first category and not nowhere dense.

33. Prove the following strong version of the open mapping theorem. *Let T be a bounded linear map from a Banach space X into a normed linear space Y. If the image $\mathscr{R}(T)$ is of the second category in Y, then:*

    a) *T is surjective (i.e. $\mathscr{R}(T)=Y$).*

    b) *T is an open mapping.*

    c) *Y is complete.*

34. Let $1\le p<q\le\infty$.

    a) Show that the inclusion map $L^q[0,1]\hookrightarrow L^p[0,1]$ is continuous, but not onto.

    b) What can be said of the category of $L^q[0,1]$ as a subspace of $L^p[0,1]$?

35. A linear operator $T:X\to Y$ is said to be *bounded below* if there exists a constant $\alpha>0$ such that

$$\|Tx\|\ge\alpha\|x\| \quad (x\in X).$$

Let T be a linear operator from a normed linear space X to a normed linear space Y. Show that the following are equivalent:

a) T is bounded below.

b) $T^{-1} : \mathscr{R}(T) \to X$ exists and is bounded.

Further, if T is a continuous linear operator from X to Y and if X is a Banach space, then each of the above equivalent assertions implies that $\overline{\mathscr{R}(T)} = \mathscr{R}(T)$.

36. Let us consider the spaces $C[0,1]$ and $C^1[0,1]$ (respectively, functions continuous and of class C^1 in the interval $[0,1]$; in the latter case we assume the existence of one-sided continuous derivatives at the interval endpoints), endowed with the sup-norm.

    a) Show that the differential operator $D : C^1[0,1] \to C[0,1]$ mapping f to $Df = f'$ has closed graph, but is unbounded.

    b) Why does not *a*) contradict the closed graph theorem?

37. Let $T : \mathscr{D}(T) \to Y$ be a bounded linear operator with domain $\mathscr{D}(T) \subset X$, where X and Y are normed spaces. Show that:

    a) If $\mathscr{D}(T)$ is a closed subset of X, then T has closed graph.

    b) If T has closed graph and Y is complete, then $\mathscr{D}(T)$ is a closed subset of X.