An introduction to the spectral theory of linear operators problem set

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1. Show that:

   a) The weak topology of a normed space is a vector space topology.

   b) The weak closure of a subspace (resp. convex set) is also a subspace (resp. convex set).

2. In a topological vector space X, a set $A \subset X$ is said to be *bounded* if it can be completely absorbed by every neighborhood of zero, that is, if to every zero neighorhood V, there corresponds $t > 0$ such that $A \subset tV$. Assume X is a normed space.

   a) Prove that weak boundedness and norm boundedness fit into this definition.

   b) Recalling that weak and norm boundedness are equivalent, show that non-trivial subspaces are (weakly, strongly) unbounded.

   c) Prove that weakly open subsets of infinite-dimensional normed spaces are unbounded.

3. Show that the weak topology of infinite-dimensional normed spaces is not metrizable.

4. Prove that in a normed space X we have $x_n \rightharpoonup x$ weakly if, and only if,

   a) the sequence $\{\|x_n\|\}_{n=1}^{\infty}$ is bounded, and

   b) for every element f of a total subset $M \subset X'$, we have $\lim_{n\to\infty} f(x_n) = f(x)$.

5. *(Hilbert space)* Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Justify the following statements about weak convergence in H.

   a) $x_n \rightharpoonup x$ weakly if, and only if, $\lim_{n\to\infty}\langle x_n, z\rangle = \langle x, z\rangle$ for all $z \in H$.

   b) The weak limit of any orthonormal sequence is 0.

   c) Let $\{u_i\}_{i\in I}$ be an orthonormal basis. Then $x_n \rightharpoonup x$ weakly if, and only if, $\{x_n\}_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty}\langle x_n, u_i\rangle = \langle x, u_i\rangle$ for all $i \in I$.

6. *(Space ℓ^p)* Justify the following statement: in the space ℓ^p, with $1 < p < \infty$, we have $x_n \rightharpoonup x$ if, and only if,

   a) the sequence $\{\|x_n\|_p\}_{n=1}^{\infty}$ is bounded, and

   b) for every fixed $j \in \mathbb{N}$, there holds $x_n(j) \to x(j)$ as $n \to \infty$, where $x_n = \{x_n(j)\}_{j=1}^{\infty}$ and $x = \{x(j)\}_{j=1}^{\infty}$.

7. Prove that ℓ^1 has the *Schur property*: a sequence in ℓ^1 converges weakly if, and only if, it converges strongly to the same limit.

8. *(Pointwise convergence)* If $x_n \in C[a,b]$ $(n \in \mathbb{N})$ and $x_n \rightharpoonup x \in C[a,b]$, show that $\{x_n\}_{n=1}^{\infty}$ is pointwise convergent on $[a,b]$, that is, $\{x_n(t)\}_{n=1}^{\infty}$ converges for every $t \in [a,b]$.

9. If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences in the same normed space X, prove that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ imply $x_n + y_n \rightharpoonup x + y$ as well as $\alpha x_n \rightharpoonup \alpha x$, where α is any scalar.

10. If $x_n \rightharpoonup x_0$ in a normed space X, show that $x_0 \in \overline{Y}$, where $Y = \operatorname{span}\{x_n : n \in \mathbb{N}\}$.

11. *a*) If M is any subspace of a normed space X, the identity

$$\overline{M} = \bigcap_{f \in M^{\perp}} \mathscr{N}(f)$$

holds, where $\overline{M}$ denotes the strong closure of M. Use this identity to deduce the Mazur theorem for subspaces, namely that the weak and the strong closures of M coincide.

b) Prove that any closed convex subset Y of a normed space X contains the limits of all weakly convergent sequences of its elements (that is, Y is *weakly sequentially closed*).

12. *(Weak Cauchy sequences)* A *weak Cauchy sequence* in a real or complex normed space X is a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that for every $f \in X'$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ or $\mathbb{C}$, respectively; note that then $\lim_{n\to\infty} f(x_n)$ exists.

a) Show that weak Cauchy sequences are bounded.

b) Let A be a set in a normed space X such that every nonempty subset of A contains a weak Cauchy sequence. Show that A is bounded.

13. *(Weak completeness)* A normed space X is said to be *weakly complete* if every weak Cauchy sequence in X converges weakly in X. Show that reflexive spaces are weakly complete.

14. Assume $f, f_1, f_2, \ldots, f_n$ are linear functionals on a vector space X. Establish the equivalence of the following statements:

a) $f \in \operatorname{span}\{f_1, f_2, \ldots, f_n\}$.

b) There exists $C \geq 0$ such that $|f(x)| \leq C \max\{|f_i(x)| : 1 \leq i \leq n\}$ $(x \in X)$.

c) f is bounded, bounded above, or bounded below, in $\bigcap_{i=1}^{n} \mathscr{N}(f_i)$.

d) $\bigcap_{i=1}^{n} \mathscr{N}(f_i) \subset \mathscr{N}(f)$.

15. Prove that in finite-dimensional spaces the strong, weak, and weak* topologies coincide.

16. Show that the dual of an infinite-dimensional normed space with the weak* topology is of the first category in itself.

17. *(Hahn-Banach separation theorem for the weak* topology)* Assume X is a normed space. Let $A \subset X'$ be a nonempty, weakly* closed and convex set, and let $x'_0 \in X' \setminus A$. Prove that there exists $x \in X$ satisfying

$$\sup\{\Re\langle x, a'\rangle : a' \in A\} < \Re\langle x, x'_0\rangle.$$

18. *(Hahn-Banach characterization of weak* closure)* Suppose X is a normed space. Let M be a subspace of X', and let $x'_0 \in X' \setminus \overline{M}^{\sigma(X',X)}$. Show that there exists $x \in X$ such that $\langle x, m'\rangle = 0$ for every $m' \in M$, but $\langle x, x'_0\rangle = 1$.

19. Prove that if an operator T on a finite-dimensional space is represented by a matrix T_E, then the adjoint operator T' is represented by the transpose of T_E.

20. *(Relation between Hilbert-adjoint and adjoint)* Let H_1 and H_2 be Hilbert spaces, and let $A_i : H_i' \to H_i$ $(i = 1, 2)$ be the corresponding Fréchet-Riesz isometric isomorphisms. Let $T : H_1 \to H_2$ be a bounded linear operator. Show that:

    a) The Hilbert space adjoint T^* and the Banach space adjoint T' of T are related by $T^* = A_1 T' A_2^{-1}$.

    b) T' is defined on the dual of the space which contains the range of T, whereas T^* is defined directly on the space which contains the range of T.

    c) For T' we have
    $$(\alpha T)' = \alpha T',$$
    but for T^* we have
    $$(\alpha T)^* = \overline{\alpha} T^*.$$

    d) In the finite dimensional case, T' is represented by the transpose of the matrix representing T, whereas T^* is represented by the complex conjugate transpose of that matrix.

21. *(Total boundedness)* Let B be a subset of a metric space X and let $\varepsilon > 0$ be given. A set $M_\varepsilon \subset X$ is called a ε*-net* for B if for every point $z \in B$ there is a point of M_ε at a distance from z less than ε. The set B is said to be *totally bounded* if for every $\varepsilon > 0$ there is a finite ε-net $M_\varepsilon \subset X$ for B, where «finite» means that M_ε is a finite set (that is, consists of finitely many points). Consequently, total boundedness of B means that for every given $\varepsilon > 0$, the set B is contained in the union of finitely many open balls of radius ε. Finally, B is said to be *relatively compact* if its closure is compact.

    Prove that:

    a) If B is relatively compact, then B is totally bounded.

    b) If B is totally bounded, then so is $\overline{B}$.

    c) If B is totally bounded, then every sequence in B has a Cauchy subsequence.

    d) If B is totally bounded and X is complete, then B is relatively compact.

    e) If B is totally bounded, then for every $\varepsilon > 0$ there exists a finite ε-net $M_\varepsilon \subset B$.

    f) If B is totally bounded, then B is separable.

22. Prove compactness of $T : \ell^2 \to \ell^2$ defined by $y = Tx$, $x = \{x(n)\}_{n=1}^{\infty}$, $y = \{y(n)\}_{n=1}^{\infty}$, $y(n) = x(n)/n$ $(n \in \mathbb{N})$.

23. Let X, Y be Banach spaces and $T : X \to Y$ be a compact linear operator. Suppose that $\{x_n\}_{n=1}^{\infty} \subset X$ is weakly convergent, say $x_n \rightharpoonup x$. Prove that $\{Tx_n\}_{n=1}^{\infty}$ is strongly convergent in Y to the limit $y = Tx$.

24. Let X and Y be Banach spaces. Prove that the range $\mathscr{R}(T)$ of a compact linear operator $T : X \to Y$ is separable.

25. Let X be a Banach space and assume $\{\lambda_1, \ldots, \lambda_n\}$ are pairwise distinct eigenvalues of an operator $T \in \mathscr{B}(X)$. Assume further that e_j is an eigenvector for λ_j $(1 \leq j \leq n)$. Show that $\{e_1, \ldots, e_n\}$ are linearly independent.

26. Let $T : X \to X$ be a compact linear operator on a Banach space X. For every $\lambda \neq 0$, the null space $\mathscr{N}(T_\lambda)$ of $T_\lambda = T - \lambda I$ is finite dimensional. Prove that actually

$$\dim \mathscr{N}\left(T_\lambda^n\right) < \infty \quad (n = 0, 1, 2, \ldots)$$

and, moreover,

$$\{0\} = \mathscr{N}\left(T_\lambda^0\right) \subset \mathscr{N}(T_\lambda) \subset \mathscr{N}\left(T_\lambda^2\right) \subset \ldots .$$

27. Let $T \in \mathscr{B}(X)$ be a compact linear operator on a Banach space X, and let $\lambda \neq 0$. It is known that the range of $T_\lambda = T - \lambda I$ is closed. Prove that, in fact, the range of $T_\lambda^n = (T - \lambda I)^n$ is closed for every $n = 0, 1, 2, \ldots$. Furthermore, show that

$$X = \mathscr{R}(T_\lambda^0) \supset \mathscr{R}(T_\lambda) \supset \mathscr{R}(T_\lambda^2) \supset \ldots .$$

28. From Exercises 26 and 27 we know that for a compact linear operator T on a Banach space X and $\lambda \neq 0$ the null spaces $\mathscr{N}\left(T_\lambda^n\right)$ are finite dimensional and satisfy $\mathscr{N}\left(T_\lambda^n\right) \subset \mathscr{N}\left(T_\lambda^{n+1}\right)$, while the ranges $\mathscr{R}(T_\lambda^n)$ are closed and satisfy $\mathscr{R}(T_\lambda^n) \supset \mathscr{R}(T_\lambda^{n+1})$ $(n = 0, 1, 2, \ldots)$. Prove that from some $n = p$ on, those null spaces are all equal; and from some $n = q$ on, those ranges are all equal, the remaining inclusions being proper. Moreover, assuming that p and q are the smallest integers with such properties, prove that $p = q$.

29. Let X, T, λ and $r = p = q$ be as in Exercise 28. Show that X can be represented in the form

$$X = \mathscr{N}\left(T_\lambda^r\right) \oplus \mathscr{R}(T_\lambda^r).$$

30. Consider the linear operator $T : \ell^2 \to \ell^2$ defined by

$$Tx = \left\{0, \frac{x(1)}{1}, \frac{x(2)}{2}, \frac{x(3)}{3}, \ldots\right\},$$

where $x = \{x(n)\}_{n=1}^\infty$. Prove that T is compact and $\sigma(T) = \{0\}$, but T has no eigenvalues.

31. Let $T : \ell^2 \to \ell^2$ be defined by $y = Tx$, $x = \{x(n)\}_{n=1}^\infty$, $y = \{y(n)\}_{n=1}^\infty$, $y(n) = \alpha(n)x(n)$ $(n \in \mathbb{N})$, where $\{\alpha(n)\}_{n=1}^\infty$ is dense on $[0, 1]$. Show that T is not compact.

32. Let $T : C[0, 1] \to C[0, 1]$ be defined by $Tx = vx$, where $v(t) = t$ $(t \in [0, 1])$. Show that T is not compact.