Hahn-Banach theorems and applications problem set

ISABEL MARRERO
Departamento de Análisis Matemático
Universidad de La Laguna
imarrero@ull.es

1. Suppose X is a vector space. All sets mentioned below are understood to be subsets of X. Prove the following statements.

   a) $2A \subset A+A$; it may happen that $2A \neq A+A$.

   b) A is convex if, and only if, $(s+t)A = sA+tA$ for all positive scalars s and t.

   c) Every union (and intersection) of balanced sets is balanced.

   d) Every intersection of convex sets is convex.

   e) If Γ is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of Γ is convex.

   f) If A and B are convex, so is $A+B$.

   g) If A and B are balanced, so is $A+B$.

   h) Parts *d*), *e*), and *f*) hold with subspaces in place of convex sets.

2. Let X, Y be vector spaces and $f: X \to Y$ a linear map. Show that:

   a) If $A \subset X$ is a subspace (or a convex set, or a balanced set), the same is true of $f(A)$.

   b) If $B \subset Y$ is a subspace (or a convex set, or a balanced set), the same is true of $f^{-1}(B)$.

3. Let X, Y be normed spaces. Show that:

   a) If $A \subset X$ and $B \subset X$, then $\overline{A}+\overline{B} \subset \overline{A+B}$.

   b) If Y is a subspace of X, so is $\overline{Y}$.

   c) If C is a convex subset of X, so are $\overline{C}$ and $\overset{\circ}{C}$.

   d) If B is a balanced subset of X, so is $\overline{B}$; if also $0 \in \overset{\circ}{B}$, then $\overset{\circ}{B}$ is balanced.

   e) If E is a bounded subset of X, so is $\overline{E}$.

4. Prove that every linear functional not identically zero on a normed space is onto and open.

5. Let X be a finite-dimensional normed space, let Y be any normed linear space, and let $T: X \to Y$ be a linear transformation. Prove that T is continuous.

6. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be finite-dimensional normed linear spaces, with $\dim X = \dim Y$. Show that X and Y are isomorphic as normed spaces.

7. Assume X is a finite dimensional normed linear space with basis $\{v_1, v_2, \ldots, v_n\}$. Prove that X' has a basis $\{f_1, f_2, \ldots, f_n\}$ such that $f_j(v_k) = \delta_{jk}$ for $1 \leq j, k \leq n$ so that, in particular, $\dim X' = \dim X$. Here,

$$\delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

   is the Kronecker delta.

8. Let X be a finite-dimensional normed space.

   a) Show that every linear functional on X is continuous.

   b) Deduce that X and X', as normed spaces, are isomorphic.

9. Let $1 \leq p < \infty$ and $1 < q \leq \infty$ be such that $1/p + 1/q = 1$. For every $y = \{y(n)\}_{n=1}^{\infty} \in \ell^q$, let $\Phi y : \ell^p \to \mathbb{K}$ be the linear functional defined by

$$(\Phi y)(x) = \sum_{n=1}^{\infty} x(n)y(n) \quad (x = \{x(n)\}_{n=1}^{\infty} \in \ell^p).$$

   Show that $\Phi y \in (\ell^p)'$, with $\|\Phi y\| = \|y\|_q$, and that the map $\Phi : \ell^q \to (\ell^p)'$ sending $y \in \ell^q$ to $\Phi y \in (\ell^p)'$ is an isometric isomorphism. Thus, for $1 \leq p < \infty$, the dual of ℓ^p «is» ℓ^q.

10. For every $y = \{y(n)\}_{n=1}^{\infty} \in \ell^1$, let $\Phi y : c_0 \to \mathbb{K}$ be the linear functional defined by

$$(\Phi y)(x) = \sum_{n=1}^{\infty} x(n)y(n) \quad (x = \{x(n)\}_{n=1}^{\infty} \in c_0).$$

   Show that $\Phi y \in (c_0)'$, with $\|\Phi y\| = \|y\|_1$, and that the map $\Phi : \ell^1 \to (c_0)'$ sending $y \in \ell^1$ to $\Phi y \in (c_0)'$ is an isometric isomorphism. Thus, the dual of c_0 «is» ℓ^1.

11. *a*) Prove that c_{00} is dense in c_0, but not in ℓ^∞.

   b) Show that for all $y = \{y(n)\}_{n=1}^{\infty} \in \ell^1$, the map

$$(\Phi y)(x) = \sum_{n=1}^{\infty} x(n)y(n) \quad (x = \{x(n)\}_{n=1}^{\infty} \in \ell^\infty)$$

   defines a continuous linear functional on ℓ^∞.

   c) Prove by counterexample that not all $f \in (\ell^\infty)'$ are of the form Φy for some $y \in \ell^1$.

12. Show that the dual of c «is» ℓ^1 by constructing explicitly an isometric isomorphism $\Phi : \ell^1 \to c'$.

13. Recall that

$$c = \left\{ x = \{x(n)\}_{n=1}^{\infty} \in \ell^\infty : \lim_{n\to\infty} x(n) \text{ exists} \right\},$$

   and consider the space

$$s = \left\{ x = \{x(n)\}_{n=1}^{\infty} \in \ell^\infty : \lim_{n\to\infty} x(2n) \text{ and } \lim_{n\to\infty} x(2n+1) \text{ exist} \right\}.$$

   Let f, f_1, f_2 be the linear functionals defined by:

$$f(x) = \lim_{n\to\infty} x(n) \quad (x = \{x(n)\}_{n=1}^{\infty} \in c),$$
$$f_1(x) = \lim_{n\to\infty} x(2n), \quad f_2(x) = \lim_{n\to\infty} x(2n+1) \qquad (x = \{x(n)\}_{n=1}^{\infty} \in s).$$

   Prove that:

a) c and s are closed subspaces of ℓ^∞, with $c \subset s$;

b) $f \in c'$ and $f_1, f_2 \in s'$, with $\|f\| = \|f_1\| = \|f_2\| = 1$; and

c) $f_1|_c = f_2|_c = f|_c$, but $f_1|_s \neq f_2|_s$.

Therefore, the equinormic extension provided by the Hahn-Banach theorem needs not be unique.

14. Let M be the subspace of ℓ^1 given by

$$M = \left\{x = \{x(k)\}_{k=1}^\infty \in \ell^1 : x(k) = 0\ (k \geq 2)\right\},$$

and consider the functionals

$$f(x) = x(1) \quad (x = \{x(k)\}_{k=1}^\infty \in M),$$
$$F(x) = \sum_{k=1}^\infty x(k), \quad F_n(x) = \sum_{k=1}^n x(k) \qquad (x = \{x(k)\}_{k=1}^\infty \in \ell^1,\ n \in \mathbb{N}).$$

Show that:

a) M is closed;

b) $f \in M'$ and $F, F_n \in (\ell^1)'$ $(n \in \mathbb{N})$, with $\|f\| = \|F\| = \|F_n\| = 1$ $(n \in \mathbb{N})$;

c) $F|_M = F_n|_M = f$ $(n \in \mathbb{N})$.

Therefore, there may be an infinity of equinormic extensions as provided by the Hahn-Banach theorem.

15. Endow $\mathbb{R}^2$ with the norm

$$\|x\|_1 = |x_1| + |x_2| \quad (x = (x_1, x_2) \in \mathbb{R}^2),$$

and consider the subspace

$$M = \{x = (x_1, 0) : x_1 \in \mathbb{R}\}$$

along with the functionals:

$$f(x) = x_1 \quad (x = (x_1, 0) \in M),$$
$$F_1(x) = x_1, \quad F_2(x) = x_1 + x_2 \quad (x = (x_1, x_2) \in \mathbb{R}^2).$$

Show that:

a) M is closed;

b) $f \in M'$ and $F_1, F_2 \in (\mathbb{R}^2)'$, with $\|f\| = \|F_1\| = \|F_2\| = 1$;

c) $F_1|_M = F_2|_M = f$.

Therefore, even in finite-dimensional spaces, the equinormic extension provided by the Hahn-Banach theorem needs not be unique.

16. *(Extensions of greater norm always exist)* Assume f is a continuous linear functional defined on the closed proper subspace M of the normed space X over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Prove that there are continuous linear extensions F of f with $\|F\| > \|f\|$.

17. *(Banach limits)* Let $\ell^\infty = \ell^\infty(\mathbb{R})$. Prove that there exists a functional $\Phi : \ell^\infty \to \mathbb{R}$ such that

    a) Φ is linear: $\Phi(x+y) = \Phi x + \Phi y$ and $\Phi(cx) = c\Phi x$ $(x, y \in \ell^\infty,\ c \in \mathbb{R})$.

    b) Φ is positive: $\Phi x \geq 0$ for every $x \in \ell^\infty$ with nonnegative terms.

    c) Φ is normalized: $\Phi(e) = 1$, where $e = \{e(n)\}_{n=1}^\infty$ with $e(n) = 1$ $(n \in \mathbb{N})$.

    d) Φ is shift-invariant: $\Phi(Sx) = \Phi x$, where $(Sx)(n) = x(n+1)$ $(n \in \mathbb{N})$.

    Further, show that the above properties of the functional Φ imply the following:

    e) Φ has norm one; in particular, $|\Phi x| \leq \|x\|_\infty$ $(x \in \ell^\infty)$.

    f) For any $x \in \ell^\infty$:

$$\liminf_{n\to\infty} x(n) \leq \Phi x \leq \limsup_{n\to\infty} x(n).$$

    g) Φ extends the limit functional defined on the subspace of convergent sequences: $\lim_{n\to\infty} x(n) = c$ implies $\Phi x = c$.

18. Suppose X is a normed space. Let A be a nonempty subset of X, and let $\{c_x : x \in A\}$ be a corresponding collection of scalars. Prove that the following are equivalent.

    a) There is a bounded linear functional f on X such that $f(x) = c_x$ for each x in A.

    b) There is a nonnegative real number M such that

$$|\alpha_1 c_{x_1} + \cdots + \alpha_n c_{x_n}| \leq M \|\alpha_1 x_1 + \cdots + \alpha_n x_n\|$$

    for each linear combination $\alpha_1 x_1 + \cdots + \alpha_n x_n$ of elements of A, that is, for each element of $\operatorname{span}\{A\}$.

    Further, show that if *b*) holds then f can be chosen in *a*) so that $\|f\| \leq M$.

19. Define in $\ell^1 = \ell^1(\mathbb{R})$ the two subspaces

$$A = \left\{x = \{x(n)\}_{n=1}^\infty \in \ell^1 : x(2n) = 0 \ (n \in \mathbb{N})\right\},$$
$$B = \left\{x = \{x(n)\}_{n=1}^\infty \in \ell^1 : x(2n) = \frac{x(2n-1)}{2^n} \ (n \in \mathbb{N})\right\}.$$

Let $p = \{p(n)\}_{n=1}^{\infty}$ be the point defined by

$$\begin{cases} p(2n) = \dfrac{1}{2^n} \\ p(2n-1) = 0 \end{cases} \qquad (n \in \mathbb{N}),$$

and set $C = A + p$. Prove that:

a) The subspaces A and B are closed, with $\overline{A+B} = \ell^1$.

b) $p \notin A + B$.

c) B and C are two convex subsets which cannot be separated by a closed hyperplane.

Therefore, it is not true that any two non-intersecting convex sets can be separated by a continuous functional, without further assumptions (such as one of the convex subsets has nonempty interior, or one is closed and the other compact).

20. Solve Exercise 19 for the ambient spaces ℓ^p $(1 < p < \infty)$ and c_0.

21. Consider the real linear space $X = c_{00}$ (sequences with compact support), and let A denote the subset of X formed by all those sequences whose last nonzero term is strictly positive, that is,

$$A = \left\{ \sum_{k=1}^{N} \alpha(k) e_k : \ \alpha(1), \alpha(2), \ldots, \alpha(N) \in \mathbb{R}, \ \alpha(N) > 0 \ (N \in \mathbb{N}) \right\},$$

where, as usual, $\{e_k\}_{k=1}^{\infty}$ denotes the set of canonical unitary sequences. Let $B = \{0\}$. Check that:

a) A, B are disjoint convex subsets of X.

b) Every nonzero linear functional on X takes on strictly positive and strictly negative values on A. Hence, no nonzero functional on X can separate A and B.

22. *(Separation of convex sets in finite-dimensional spaces)* Let A and B denote disjoint, nonempty, convex subsets of $\mathbb{R}^N$. Prove that there exists a linear functional on $\mathbb{R}^N$ which separates A and B. More precisely, there exist $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, such that

$$\sum_{k=1}^{N} \alpha_k a_k \leq \gamma \leq \sum_{k=1}^{N} \alpha_k b_k$$

for any $(a_1, a_2, \ldots, a_N) \in A$ and $(b_1, b_2, \ldots, b_N) \in B$.

23. Assume A and B are disjoint, nonempty, convex sets in a normed vector space X. Give a direct proof of the following statements.

a) If A is open, then there exist $\Lambda \in X'$ and $\gamma \in \mathbb{R}$ such that $\Re \Lambda a < \gamma \leq \Re \Lambda b$ for every $a \in A$ and every $b \in B$.

b) If A is compact and B is closed, then there exist $\Lambda \in X'$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\Re \Lambda a < \gamma_1 < \gamma_2 < \Re \Lambda b$ for every $a \in A$ and every $b \in B$.

24. *(Mazur separation theorem)* Let X be a normed space, N a linear subspace of X, and G a convex open subset of X such that $N \cap G = \emptyset$. Show that there exists a closed hyperplane H of X such that $N \subset H$ and $H \cap G = \emptyset$.

25. Suppose B is a convex, balanced, closed set in a normed space X, and let $x_0 \in X \setminus B$. Prove that there exists $\Lambda \in X'$ such that $|\Lambda x| \leq 1$ for all $x \in B$, but $\Lambda x_0 > 1$.

26. Let $\{x_i\}_{i=1}^n$ be a linearly independent set of vectors in a normed space X and $\{\alpha_i\}_{i=1}^n$ be a set of real numbers. Show that there exists $f \in X'$ such that $f(x_i) = \alpha_i$ for $i = 1, \ldots, n$.

27. Let X be a normed space. Prove that if X' is separable, then so is X.

28. *(Adjoint operator)* Let X and Y be Banach spaces and $T \in \mathscr{B}(X,Y)$. The adjoint of T is defined as an operator $T' : Y' \to X'$ by

$$\left(T'g\right)(x) = g(Tx) \quad (g \in Y',\, x \in X).$$

Show that $T' \in \mathscr{B}(Y', X')$, with $\|T'\| = \|T\|$.

29. *(Dual of a subspace)* Let X be a normed space with dual X', M a subspace of X, and

$$M^\perp = \{f \in X' : f(x) = 0 \; (x \in M)\}$$

its *annihilator*. Prove that $M^\perp$ is a closed subspace of X', and that the map $\Phi : X'/M^\perp \to M'$ given by

$$\Phi(f + M^\perp) = f|_M \quad (f + M^\perp \in X'/M^\perp)$$

is an isometric isomorphism.

30. *(Dual of a quotient space)* Let M be a closed subspace of a normed space X, and let $M^\perp$ denote its annihilator. Consider the normed quotient space X/M and the quotient canonical projection $\pi : X \to X/M$, defined by $\pi(x) = x + M$ $(x \in X)$. Show that the map $\Psi : (X/M)' \to M^\perp$ given by

$$\Psi g = g \circ \pi \quad (g \in (X/M)')$$

is an isometric isomorphism.

31. *(Best approximation in dual space)* Let X be a normed space with dual X', M a subspace of X, and $M^\perp$ its annihilator. Let $f \in X'$. Prove that

$$\min\{\|f - h\| : h \in M^\perp\} = \sup\{|f(m)| : m \in M,\ \|m\| = 1\}.$$

32. *(Closure of a subspace)* Let X be a normed space, M a subspace of X, and $x_0 \in X \setminus \overline{M}$, so that $d(x_0, M) > 0$.

    a) Show that there exists $f \in M^\perp$ such that $\|f\| = 1$ and $f(x_0) = d(x_0, M)$.

b) Deduce that

$$\overline{M} = \bigcap_{f \in M^\perp} \mathcal{N}(f).$$

33. *(Dense subspace, 1)* Let X, Y be normed spaces, and let M be a subspace of X.

    a) Prove that $\overline{M} = X$ if, and only if, $M^\perp = \{0\}$.

    b) Let $T \in \mathscr{B}(X,Y)$. Show that T' is injective if, and only if, $\mathscr{R}(T)$ is dense in Y.

34. *(Dense subspace, 2)* Let M be a subspace of a normed space X. The *preannihilator* ${}^\perp N$ of a subspace $N \subset X'$ is defined as

$$^\perp N = \{x \in X : f(x) = 0 \ (f \in N)\}.$$

    Show that:

    a) $\overline{M} = {}^\perp\left(M^\perp\right)$.

    b) M is dense in X if, and only if, $M^\perp = \{0\}$.

35. Let $(X, \|\cdot\|)$ be a normed space. Associate to each $x \in X$ the linear functional $\widehat{x} : X' \to \mathbb{K}$ defined by $\widehat{x}(f) = f(x)$. Show that the so-called *canonical embedding* $J : X \to X'' = (X')'$ defined by $Jx = \widehat{x}$ is, in fact, a linear isometry from X onto $J(X)$, by proving that:

    a) $\widehat{x}$ is linear.

    b) $|\widehat{x}(f)| \leq \|x\| \, \|f\| \ (f \in X')$; thus $\widehat{x} \in X''$, with $\|\widehat{x}\| \leq \|x\|$.

    c) $\|\widehat{x}\| = \|x\| \ (x \in X)$.

36. Deduce from Exercise 35 that every normed space admits a completion.

37. A normed space X is said to be *reflexive* if the canonical embedding J is onto. Show that ℓ^p $(1 < p < \infty)$ is reflexive, but ℓ^1 and ℓ^∞ are not. [*Remark:* Note that only Banach spaces can be reflexive, since X'' is automatically Banach and reflexivity means there is a Banach space isomorphism between X and X''.]

38. Prove that every finite-dimensional normed space is reflexive.

39. Show that reflexive spaces are, in a sense, an intermediate class between Hilbert and Banach spaces, by proving the following statements.

    a) If a normed space is reflexive, then it is a Banach space.

    b) Every Hilbert space is reflexive.

40. Show that a Banach space X is reflexive if, and only if, X' is reflexive.

41. Let X be a Banach space.

a) Prove that X is reflexive and separable if, and only if, X' is reflexive and separable.

b) Show by counterexample that X' needs not be separable when X is separable but non-reflexive.

[*Hint:* Cf. Exercises 27, 37, and 40.]