Basic theory of Hilbert spaces problem set

ISABEL MARRERO
Departamento de Análisis Matemático
Universidad de La Laguna
imarrero@ull.es

1. Assume X is a real inner product space. Show that $x, y \in X$ and $\|x\| = \|y\|$ implies

$$\langle x+y, x-y\rangle = 0.$$

   In the case $X = \mathbb{R}^2$, this is a well-known geometric statement – which one?

2. Let X be a real inner product space. Show that if $x, y \in X$ satisfy $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, then $x \perp y$. Does this result hold true in complex inner product spaces?

3. Let x and y belong to a pre-Hilbert space. Prove the equivalence of the following statements:

   a) $x \perp y$;

   b) $\|x+\alpha y\| = \|x-\alpha y\|$ $(\alpha \in \mathbb{K})$;

   c) $\|x+\alpha y\| \geq \|x\|$ $(\alpha \in \mathbb{K})$.

4. Let X be an inner product space. Prove the following statements:

   a) If $u, v \in X$ and $\langle x, u\rangle = \langle x, v\rangle$ $(x \in X)$, then $u = v$.

   b) If $\{x_n\}_{n=1}^{\infty} \subset X$, $x \in X$ are such that $\lim_{n\to\infty} \|x_n\| = \|x\|$ and $\lim_{n\to\infty}\langle x_n, x\rangle = \langle x, x\rangle$, then $\lim_{n\to\infty} x_n = x$.

   c) If $\{x_n\}_{n=1}^{\infty} \subset X$ and $x = \sum_{n=1}^{\infty} x_n$, then

$$\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x_n, y\rangle \quad (y \in X).$$

5. Let X be an inner product space. Prove the *Apollonius identity*

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{1}{2}(x+y)\right\|^2 \quad (x, y, z \in X),$$

   a) directly;

   b) using the parallelogram law.

6. Let $[a, b]$ be a finite interval.

   a) Apply the Cauchy-Schwarz inequality to prove that $L^2[a, b] \subset L^1[a, b]$.

   b) Show by counterexample that this inclusion is strict.

7. Prove that all norms in a finite-dimensional linear space are equivalent, although not all of them are induced by an inner product.

8. Show that the sup-norm on $C[a, b]$ is not induced by an inner product.

9. Prove Theorem 2.15 from the lecture notes: *Given an inner product space $(X, \langle\cdot,\cdot\rangle)$, there exist a Hilbert space H and an unitary isomorphism $T : X \to T(X) \subset H$ such that $T(X)$ is dense in H. The space H is unique up to unitary isomorphisms.* [*Hint*: Use the fact that every metric space admits a completion. For another proof, see Exercise 34.]

10. Let X_i $(i \in \mathbb{N},\ 1 \leq i \leq k)$ be linear spaces equipped with inner products $\langle \cdot, \cdot \rangle_i$, respectively. The product space $X_1 \times X_2 \times \cdots \times X_k = \prod_{i=1}^k X_i$ is defined by

$$\prod_{i=1}^{k} X_i = \{(x_1, x_2, \ldots, x_k) : x_i \in X_i,\ i \in \mathbb{N},\ 1 \leq i \leq k\}.$$

In $\prod_{i=1}^k X_i$ we consider coordinatewise addition:

$$(x_1, x_2, \ldots, x_k) + (y_1, y_2, \ldots, y_k) = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k)$$

and coordinatewise scalar multiplication:

$$\lambda\,(x_1, x_2, \ldots, x_k) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_k).$$

Show that an inner product can be defined in $\prod_{i=1}^k X_i$ by setting

$$\langle (x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k) \rangle = \sum_{i=1}^{k} \langle x_i, y_i \rangle_i,$$

and that $\prod_{i=1}^k X_i$ with this inner product is a Hilbert space if so are X_i $(i \in \mathbb{N},\ 1 \leq i \leq k)$.

11. Let X be the pre-Hilbert space consisting of the polynomial $x = 0$ along with all real polynomials in the variable $t \in [a,b]$ with degree less than or equal to 2, endowed with the scalar product

$$\langle x, y \rangle = \int_a^b x(t)\overline{y(t)}\, dt \quad (x, y \in X).$$

   a) Show that X is complete.

   b) Let $Y = \{x \in X : x(a) = 0\}$. Is Y a subspace of X?

   c) Consider the set of all $x \in X$ of degree 2. Is this set a subspace of X?

12. Let

$$M = \left\{ f \in C[0,1] : \int_0^{1/2} f(t)\, dt - \int_{1/2}^{1} f(t)\, dt = 1 \right\}.$$

Prove that M is a convex closed set of $(C[0,1], \|\cdot\|_\infty)$ with no elements of minimal norm. Does this fact contradict the minimizing vector theorem?

13. Let

$$M = \left\{ f \in L^1[0,1] : \int_0^1 f(t)\, dt = 1 \right\}.$$

Show that M is a convex closed set of $(L^1[0,1], \|\cdot\|_1)$ with an infinity of minimal norm elements. Does this fact contradict the minimizing vector theorem?

14. Let $\{x_n\}_{n=1}^{\infty}$ be an orthogonal sequence in a Hilbert space H, satisfying

$$\sum_{n=1}^{\infty} \|x_n\|^2 < \infty.$$

Show that the series $\sum_{n=1}^{\infty} x_n$ converges in H. Is this still true if the orthogonality assumption is dropped?

15. Let H be a Hilbert space. Show that

$$\|x-z\| = \|x-y\| + \|y-z\| \quad (x,y,z \in H)$$

if, and only if, $y = \alpha x + (1-\alpha)z$ for some $\alpha \in [0,1]$.

16. Let H be a Hilbert space and let P and Q denote the orthogonal projections on the closed subspaces M and N, respectively. Show that if $M \perp N$, then $P+Q$ is the orthogonal projection on $M \oplus N$.

17. Let P and Q denote orthogonal projections in a Hilbert space. Assuming $PQ = QP$, show that $P+Q-PQ$ is an orthogonal projection and find its range.

18. Let

$$M = \left\{ f \in L^2[0,2\pi] : \int_0^{2\pi} f(x)\,dx = 0 \right\}.$$

Assuming we are placed at the point $g(x) = 3\cos^2 5x$, what is the length of the shortest path reaching M?

19. Let M denote a closed convex subset of the Hilbert space H, and let $y_0 \in M$, $x \in H$. Prove that

$$\|x-y_0\| = \min\{\|x-y\| : y \in M\}$$

if, and only if,

$$\Re\langle x-y_0, y-y_0\rangle \le 0 \quad (y \in M).$$

Here, as usual, $\Re z$ denotes the real part of $z \in \mathbb{C}$.

20. Let H be a Hilbert space, and let M be a closed subspace of H. Show that

$$\min\{\|x-x_0\| : x \in M\} = \max\left\{ |\langle x_0, y\rangle| : y \in M^{\perp},\ \|y\| = 1 \right\}.$$

21. Compute

$$\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2\,dx$$

and find

$$\max_{g} \int_{-1}^{1} x^3 g(x)\,dx,$$

where g is subject to the following conditions:

$$\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} xg(x)\,dx = \int_{-1}^{1} x^2 g(x)\,dx = 0, \qquad \int_{-1}^{1} |g(x)|^2\,dx = 1.$$

22. Let H be a Hilbert space, and let M be a nonempty subset of H.

    a) Prove that M is total in H if, and only if, $M^{\perp} = \{0\}$.

    b) Assuming M is a subspace, prove that M is dense in H if, and only if, $M^{\perp} = \{0\}$.

    c) Show by counterexample that the subspace assumption in *b*) cannot be dropped.

    d) Conclude that, in a Hilbert space, every dense set is total, but not every total set is dense.

23. Consider

$$M = \left\{ x = \{x(n)\}_{n=1}^{\infty} \in \ell^2 : \sum_{n=1}^{\infty} x(n) = 0 \right\}.$$

    Prove the following statements:

    a) M is a subspace of ℓ^2.

    b) M is dense in ℓ^2.

    c) $M + M^{\perp} \neq \ell^2$. Does this fact contradict the orthogonal projection theorem?

24. *a*) Find the Fourier series (with respect to the usual trigonometric orthonormal system) for the function

$$f(x) = |x| \quad (x \in [-\pi, \pi]).$$

    b) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

25. *a*) Find the Fourier series (with respect to the usual trigonometric orthonormal system) for the function

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ \cos x, & 0 \le x < \pi. \end{cases}$$

    b) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2}$.

26. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $L^2[0,1]$. Construct from it an orthonormal basis for $L^2(I)$, where I is any finite interval.

27. *a*) *(Gram-Schmidt orthonormalization process)* Let $\{x_n : n = 1,2,3,\dots\}$ be a linearly independent set of vectors in a Hilbert space H. Put $u_1 = x_1/\|x_1\|$. Having obtained $u_1,\dots,u_{n-1}$, define

$$v_n = x_n - \sum_{i=1}^{n-1} \langle x_n, u_i \rangle u_i, \qquad u_n = \frac{v_n}{\|v_n\|}.$$

Show that this construction yields an orthonormal set $\{u_n\}_{n=1}^{\infty}$ such that $\{x_1,\dots,x_N\}$ and $\{u_1,\dots,u_N\}$ have the same span for all $N \in \mathbb{N}$.

b) A metric space is *separable* if it contains a countable dense subset. Use the above ortonormalization process to prove the existence of a maximal orthonormal set in separable Hilbert spaces without appealing to transfinite induction.

28. Show that the functions

$$x_0(t) = \frac{1}{2\pi}, \quad x_{2n-1}(t) = \frac{1}{\pi}\cos(nt), \quad x_{2n}(t) = \frac{1}{\pi}\,\mathrm{sen}(nt) \quad (n \in \mathbb{N})$$

form an orthogonal sequence in $L^2[-\pi,\pi]$. Is this sequence orthonormal?

29. Consider in $L^2[0,1]$ the sequence of *Rademacher functions*:

$$e_n(t) = \sum_{j=0}^{2^n-1} (-1)^j \chi_{]\frac{j}{2^n},\frac{j+1}{2^n}[}(t) \quad (n \in \mathbb{N}).$$

a) Draw the graphs of e_1, e_2, e_3 and e_4.

b) Show that $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence in $L^2[0,1]$.

c) Show that $\{e_n\}_{n=1}^{\infty}$ is not an orthonormal basis.

30. Consider in $L^2[0,1]$ the sequence of *Haar functions*:

$$h_1(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h_{2^m+k}(t) = \begin{cases} \sqrt{2^m}, & \frac{k-1}{2^m} \le t < \frac{2k-1}{2^{m+1}} \\ -\sqrt{2^m}, & \frac{2k-1}{2^{m+1}} \le t < \frac{k}{2^m} \\ 0, & \text{otherwise,} \end{cases}$$

where $k = 1,2,\dots,2^m$, $m \in \mathbb{N}_0$.

a) Sketch the graphs of $h_1, h_2, \dots, h_8$.

b) Show that $\{h_n\}_{n=1}^{\infty}$ is an orthonormal sequence in $L^2[0,1]$.

c) Show that $\{h_n\}_{n=1}^{\infty}$ is an orthonormal basis in $L^2[0,1]$.

31. *a*) Let $f \neq 0$ be a linear functional on a vector space E, and let N denote the kernel of f. Prove that there exists a $y \in E$ with the following property: every $x \in E$ can be uniquely represented as $x = \lambda y + z$, where $z \in N$ and λ is an appropriate scalar.

b) Deduce that two linear functionals with the same kernel are scalar multiples of each other.

32. Let X be a pre-Hilbert space. Prove the following:

    a) The map $T: X \to X'$ defined by $(Ty)(x) = \langle x, y \rangle$ $(x, y \in X)$ is conjugate linear and isometric.

    b) T is onto if, and only if, X is Hilbert.

33. Show that the dual X' of a pre-Hilbert space X is a Hilbert space.

34. *a*) Given a pre-Hilbert space X, denote by $X'' = (X')'$ its *bidual*. Show that the map $\Phi: X \to X''$ defined by $(\Phi x)f = f(x)$ $(x \in X,\ f \in X')$ is a unitary isomorphism from X onto $\mathscr{R}(\Phi) \subset X''$, with $\mathscr{R}(\Phi)$ dense in X''.

    b) In the above notation, prove that if X is a Hilbert space then Φ is onto (in other words, every Hilbert space is *reflexive*). Conversely, show that if Φ is onto then X is a Hilbert space.

    c) Deduce from part *a*) that every pre-Hilbert space admits a *completion*. That is, given a pre-Hilbert space X, there is a Hilbert space H and a unitary isomorphism $T: X \to \mathscr{R}(T) \subset H$ such that $\mathscr{R}(T)$ is dense in H. Prove that H is unique except for unitary isomorphisms. [Cf. Exercise 9.]

35. Show that if X is a pre-Hilbert space with the property that $M^{\perp\perp} = M$ for every closed subspace M, then X is a Hilbert space. [*Hint*: Use Exercise 32 and review the proof of the Fréchet-Riesz representation theorem.]

36. Let f be a linear functional on a Hilbert space, and let N denote the kernel of f.

    a) Prove that if f is not continuous, then N is a dense subspace.

    b) Deduce that f is continuous if, and only if, N is a closed subspace.

37. Using the Fréchet-Riesz representation theorem, show that if M is a complete subspace of a pre-Hilbert space X, then $X = M \oplus M^{\perp}$.

38. Let H be a Hilbert space and $f \neq 0$ a continuous linear functional on H, with kernel N. Prove that $N^{\perp}$ is a one-dimensional subspace of H.

39. Show that the dual of ℓ^2, as a real vector space, is ℓ^2.

40. Find the Fréchet-Riesz representation of the functionals defined on ℓ^2 by:

    a) $f(x) = x(3) + x(4)$;

    b) $g(x) = \sum_{n=1}^{\infty} x(n)$.

41. Let

$$\begin{array}{rccl} \varphi: & \ell^2 & \longrightarrow & \mathbb{C} \\ & v & \longmapsto & \varphi(v) = \displaystyle\sum_{n=1}^{\infty} \frac{v(n)}{2^{n-1}}. \end{array}$$

Prove that φ is a continuous linear functional on ℓ^2, find its Fréchet-Riesz representation, and compute $\|\varphi\|$.

42. Let

$$\begin{array}{rccl} \varphi: & L^2(\mathbb{R}) & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & \varphi(f) = \int_{-1}^{1} 3xf(x)\,dx. \end{array}$$

Prove that φ is a continuous linear functional on $L^2(\mathbb{R})$, find its Fréchet-Riesz representation, and compute $\|\varphi\|$.

43. Use the Fréchet-Riesz representation theorem to define an inner product on the dual of a Hilbert space, and prove that the norm associated with such a product coincides with the standard operator norm.

44. Assume E, F are linear spaces and $\varphi : E \times E \to F$ is a bilinear map. Show that

$$\varphi(x+y, x+y) - \varphi(x-y, x-y) + i\,[\varphi(x+iy, x+iy) - \varphi(x-iy, x-iy)] = 0 \quad (x, y \in E).$$

45. Let E, F, G be normed spaces and $\varphi : E \times F \to G$ a bilinear map. Prove the equivalence of the following statements:

a) φ is bounded.

b) If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} \varphi(x_n, y_n) = \varphi(x, y)$.

c) If $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty} \varphi(x_n, y_n) = 0$.

46. Let E, F be normed spaces and let B be a Banach space. Suppose that M is a dense subspace of E, N is a dense subspace of F, and $\varphi : M \times N \to B$ is a bounded bilinear map. Prove that there exists a unique bounded bilinear map $\phi : E \times F \to B$ such that $\phi(x, y) = \varphi(x, y)$ $(x \in M,\ y \in N)$.

47. Let E, F, G be normed spaces and $\varphi : E \times F \to G$ be a bilinear map, continuous in each variable. Prove that if E or F are Banach spaces, then φ is bounded.

48. Let E, F be linear spaces and $\varphi : E \times E \to F$ a bilinear or sesquilinear map. Prove that φ satisfies the «parallelogram law»:

$$\varphi(x+y, x+y) + \varphi(x-y, x-y) = 2\varphi(x, x) + 2\varphi(y, y) \quad (x, y \in E).$$

49. Assume X is a pre-Hilbert space, G is a normed space, and $\varphi : X \times X \to G$ is a bounded bilinear map with the property that $\varphi(y, x) = \varphi(x, y)$ $(x, y \in X)$. Show that $\|\varphi\| = \sup\{\|\varphi(x, x)\| : x \in X,\ \|x\| \le 1\}$.

50. Prove that the inner product of a pre-Hilbert space X is a bounded sesquilinear form on X, and find its norm.

51. A *seminorm* on the linear space E is a map $p : E \to \mathbb{R}$ which satisfies the following axioms:

(i) $p(x) \ge 0$ $(x \in E)$;

(ii) $p(\lambda x) = |\lambda| p(x)$ $(\lambda \in \mathbb{K},\ x \in E)$;

(iii) $p(x+y) \le p(x) + p(y)$ $(x, y \in E)$.

Assume that φ is a positive sesquilinear form on a linear space E. Show that $p(x) = \varphi(x,x)^{1/2}$ $(x \in E)$ defines a seminorm on E.

52. Let X, Y be pre-Hilbert spaces, and let X^*, Y^* be their complex conjugate pre-Hilbert spaces. Prove that the following are equivalent:

    a) $T : X \to Y$ is linear conjugate, and $\langle Tx, Ty \rangle = \langle y, x \rangle$ $(x, y \in X)$.

    b) $T : X \to Y^*$ is linear, and $[Tx, Ty] = \langle x, y \rangle$ $(x, y \in X)$.

    c) $T : X^* \to Y$ is linear, and $\langle Tx, Ty \rangle = [x, y]$ $(x, y \in X^*)$.

    [*Remark*: If $X = Y$, a surjective operator T satisfiying *a*) is called a *conjugation* of the pre-Hilbert space X. For instance, $(Tx)(n) = \overline{x(n)}$ $(x \in \ell^2,\ n \in \mathbb{N})$ defines a conjugation of the Hilbert space ℓ^2.]

53. Let H be a Hilbert space. Show that:

    a) The operator $U : H \to (H')^*$ which maps $x \in H$ to the continuous linear functional $Ux = x'$ defined by x on H is a linear isomorphism.

    b) Endowed with the inner product $[x', y'] = \langle y', x' \rangle$ $(x', y' \in (H')^*)$, $(H')^*$ is a Hilbert space.

    c) The map $U : H \to (H')^*$ is a unitary isomorphism.

    [*Hint*: Cf. Exercise 33. The operator $T : H \to H'$ defined by $Tx = x'$ $(x \in H)$ satisfies condition *a*) in Exercise 52.]

54. Let X, Y be pre-Hilbert spaces, and let the map $T : X \to Y$ be such that $\langle Tu, Tv \rangle = \langle v, u \rangle$ $(u, v \in X)$. Prove the following:

    a) If $\mathscr{R}(T)$ is a subspace of Y, then T is conjugate linear.

    b) In particular, if $X = Y$ and T is onto, then T is a conjugation of X (cf. Exercise 52).

    c) If $X = H$ is a Hilbert space and $\mathscr{R}(T)$ is a total subset of Y, then T is onto.

55. Let S be a bounded linear operator on a Hilbert space, such that $S^*S = 0$. Show that $S = 0$.

56. Prove that any two linear operators S, T on a Hilbert space satisfy the «polarization identity»:

$$4T^*S = (S+T)^*(S+T) - (S-T)^*(S-T) + i[(S+iT)^*(S+iT) - (S-iT)^*(S-iT)].$$

57. Let H be a Hilbert space. Prove that a linear operator P on H is idempotent and self-adjoint if, and only if, P is the orthogonal projection onto $\mathscr{R}(P)$, that is, $H = \mathscr{N}(P) \oplus \mathscr{R}(P)$, with $\mathscr{N}(P) \perp \mathscr{R}(P)$.

58. Let H be a Hilbert space, M a closed subspace of H and T a bounded linear operator on H. The subspace M is said to be *invariant* under T provided that $Tx \in M$ for all $x \in M$. Moreover, M is said to *reduce* T whenever M and $M^\perp$ are invariant under T. Let P be the orthogonal projection onto M.

a) Prove that M is invariant under T if, and only if, $PTP = TP$.

b) Prove the equivalence of the following statements:

i) M reduces T.

ii) $PT = TP$.

iii) M is invariant under T and T^*.

59. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis in a Hilbert space H, and let $\{\mu_k\}_{k=1}^{\infty}$ be a bounded sequence of complex numbers, with $M = \sup\{|\mu_k| : k \in \mathbb{N}\}$.

a) Show that there exist a unique bounded linear operator T on H with $Te_k = \mu_k e_k$ $(k \in \mathbb{N})$.

b) Show that $\|T\| = M$.

c) Find T^* and compute its norm.

d) Prove that T is *normal*, that is, $TT^* = T^*T$.

e) What property must satisfy the sequence $\{\mu_k\}_{k=1}^{\infty}$ for T to be self-adjoint?

f) Assuming that $|\mu_k| > 1$ $(k \in \mathbb{N})$, prove that there exists T^{-1} as a bounded linear operator on H.